DECOMPOSITIONS OF COMPLETE MULTIPARTITE GRAPHS INTO COMPLETE GRAPHS

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ABSTRACT. Let $k \geq \ell \geq 1$ and $n \geq 1$ be integers. Let $G(k, n)$ be the complete k-partite graph with n vertices in each colour class. An ℓ-*decomposition* of $G(k, n)$ is a set X of copies of K_k in $G(k, n)$ such that each copy of K_ℓ in $G(k, n)$ is a subgraph of exactly one copy of K_k in X. This paper asks: when does $G(k, n)$ have an ℓ-decomposition? The answer is well known for the $\ell = 2$ case. In particular, $G(k, n)$ has a 2-decomposition if and only if there exists $k - 2$ mutually orthogonal Latin squares of order n. For general ℓ, we prove that $G(k, n)$ has an ℓ-decomposition if and only if there are $k - \ell$ Latin cubes of dimension ℓ and order n, with an additional property that we call mutually invertible. This property is stronger than being mutually orthogonal. An ℓ-decomposition of $G(k, n)$ is then constructed whenever no prime less than k divides n.

1. INTRODUCTION

Let $G(k, n)$ be the complete k-partite graph with n vertices in each colour class. Formally, $G(k, n)$ has vertex set $[k] \times [n]$ where (c, u) is adjacent to (d, v) if and only if $c \neq d$. Here $[n] := \{1, 2, \ldots, n\}$. Sometimes we use a vector $(v_1, \ldots, v_k)$ to denote the clique with vertex set $\{(i, v_i) : i \in [k]\}$.

For $k \geq \ell \geq 2$, an ℓ-*decomposition* of $G(k, n)$ is a set X of copies of K_k in $G(k, n)$, such that each copy of K_ℓ in $G(k, n)$ is a subgraph of exactly one copy of K_k in X. Here K_k is the complete graph on k vertices. This paper considers the question:

When does $G(k, n)$ have an ℓ-decomposition?

First note that every ℓ-decomposition of $G(k, n)$ contains exactly n^ℓ copies of K_k (since K_k contains $\binom{k}{\ell}$ copies of K_ℓ, and $G(k, n)$ contains $\binom{k}{\ell} n^\ell$ copies of K_ℓ).

The $\ell = 2$ case of our question corresponds to a proper partition of the edge-set of $G(k, n)$, called a 'decomposition'. It is well known that this case can be answered in terms of the existence of mutually orthogonal Latin squares (Theorem 1). These connections are explored in Section 2.

Given this relationship, it is natural to consider the relationship between ℓ-decompositions and mutually orthogonal Latin cubes, which are a higher dimensional analogue of Latin

squares. However, the situation is not as simple as the $\ell = 2$ case. The first contribution of this paper is a characterisation of ℓ-decompositions in terms of Latin cubes of dimension ℓ, with an additional property that we call mutually invertible (Theorem 7). This property is stronger than being mutually orthogonal. For $\ell = 2$ these two properties are equivalent. These results are presented in Section 3.

Then in Section 4, we construct an ℓ-decomposition whenever no prime less than k divides n (Theorem 10). Finally we relax the definition of ℓ-decomposition to allow each K_ℓ to appear in *at least* one copy of K_k. Results are obtained for all n (Theorem 13).

2. LATIN SQUARES AND THE $\ell = 2$ CASE

A *Latin square* of order n is an $n \times n$ array in which each row and each column is a permutation of $[n]$. Two Latin squares are *orthogonal* if superimposing them produces each element of $[n] \times [n]$ exactly once. Two or more Latin squares are mutually orthogonal (MOLS) if each pair is orthogonal. If $L_1, \ldots, L_{k-2}$ are mutually orthogonal Latin squares of order n, then it is easily verified that the n^2 copies of K_k defined by the vectors

$$(1) \qquad (L_1(x, y), \ldots, L_{k-2}(x, y), x, y) \ ,$$

where $(x, y) \in [n]^2$, form an edge-partition of $G(k, n)$. In fact, the following well-known converse result holds; see [1, page 162].

Theorem 1. $G(k, n)$ *has a 2-decomposition if and only if there exists $k - 2$ mutually orthogonal Latin squares of order n.*

There are at most $n - 1$ MOLS of order n; see [1, page 162]. On the other hand, MacNeish [20] proved that if p is the least prime factor of n then there exists $p - 1$ MOLS of order n. With Theorem 1 this implies:

Proposition 2. *If p is the least prime factor of n and $k = p + 1$, then there exists an edge-partition of $G(k, n)$ into n^2 copies of K_k.*

Bose, Shrikhande and Parker [8, 9] proved that for all n except 2 and 6 there exists a pair of MOLS of order n. With Theorem 1 this implies:

Proposition 3. *For all n except 2 and 6 there is an edge-partition of $G(4, n)$ into n^2 copies of K_4.*

Other values of k and n for which there is a 2-decomposition of $G(k, n)$ are immediately obtained by applying Theorem 1 with known results about the existence of MOLS; see [1].

3. Latin Cubes

A *d-dimensional Latin cube of order n* is a function $L : [n]^d \to [n]$ such that each row is a permutation of $[n]$; that is, for all $i \in [d]$ and $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d \in [n]$,

$$\{L(x_1, \ldots, x_{i-1}, j, x_{i+1}) : j \in [n]\} = [n] \ .$$

If $L_1, \ldots, L_d$ are d-dimensional Latin cubes of order n, and for every $(v_1, \ldots, v_d) \in [n]^d$ there exists $x_1, \ldots, x_d$ such that $L_i(x_1, \ldots, x_d) = v_i$ for all $i \in [d]$, then $L_1, \ldots, L_d$ are said to be *orthogonal*. Thus superimposing $L_1, \ldots, L_d$ produces each element of $[n]^d$ exactly once. If every d-tuple of a set $\mathcal{L}$ of d-dimensional Latin cubes of order n are orthogonal then $\mathcal{L}$ is *mutually orthogonal*. For results on mutually orthogonal Latin cubes and related concepts see [2–6, 19, 21, 22].

From an ℓ-decomposition of $G(k, n)$, it is possible to construct a set of $k - \ell$ mutually orthogonal ℓ-dimensional Latin cubes (see Theorem 7). However, the natural analogue of (1) does not hold. Consider the following set $\{L_1, L_2, L_3\}$ of three mutually orthogonal 3-dimensional Latin cubes of order 4.

111	233	344	422		222	144	433	311		333	411	122	244		444	322	211	133
343	421	112	234		434	312	221	143		121	243	334	412		212	134	443	321
424	342	231	113		313	431	142	224		242	124	413	331		131	213	324	442
232	114	423	341		141	223	314	432		414	332	241	123		323	441	132	214

In this example, the Latin cubes are superimposed so that L_1 is:

1	2	3	4		2	1	4	3		3	4	1	2		4	3	2	1
3	4	1	2		4	3	2	1		1	2	3	4		2	1	4	3
4	3	2	1		3	4	1	2		2	1	4	3		1	2	3	4
2	1	4	3		1	2	3	4		4	3	2	1		3	4	1	2

The natural analogue of (1) would be to construct copies of K_6 in $G(6, 4)$ of the form

$$(L_1(x, y, z), L_2(x, y, z), L_3(x, y, z), x, y, z) \ ,$$

where $x, y, z \in [4]$. However, in this case not every copy of K_3 in $G(6, 4)$ is covered. For example, $\{(1, 1), (2, 2), (6, 2)\}$ is not covered (since $z = 2$ and $L_1(x, y, 2) = 1$ implies $L_2(x, y, 2) = 4$, as shown by the underlined entries above).

Below we introduce a stronger condition than orthogonality so that this construction does provide an ℓ-decomposition.

We consider k-tuples in $[n]^k$ to be functions from $[k]$ to $[n]$. So that for $t := (t_1, \ldots, t_k)$, we use the notation $t(i) = t_i$. A set X of k-tuples in $[n]^k$ is said to be *ℓ-extendable* if for all indices $s_1 < s_2 < \cdots < s_\ell$ (where $s_i \in [k]$) and for every element $(x_1, \ldots, x_\ell) \in [n]^\ell$, there exists a unique $t \in X$ such that $t(s_i) = x_i$ for all $i \in [\ell]$.

Lemma 4. *Let X be an ℓ-extendable set of k-tuples in $[n]^k$, and let $s_1 < s_2 < \cdots < s_\ell$, where $s_i \in [k]$. Let t be the unique k-tuple such that $t(s_i) = x_i$ for all $i \in [\ell]$. For every $j \in [k] \setminus \{s_1, s_2, \ldots, s_\ell\}$, let L_j be the function defined by $L_j(x_1, \ldots, x_\ell) := t(j)$. Then L_j is an ℓ-dimensional Latin cube.*

Proof. Let $(x_1, \ldots, x_\ell) \in [n]^\ell$ and $h \in [\ell]$. Suppose that for some $x'_h \in [n]$,

$$L_j(x_1, \ldots, x_{h-1}, x_h, x_{h+1}, \ldots, x_\ell) = y = L_j(x_1, \ldots, x_{h-1}, x'_h, x_{h+1}, \ldots, x_\ell) \ .$$

Then there is a tuple t' in X such that $t'(s_i) = x_i$ for $s_i \in \{s_1, \ldots, s_\ell\} \setminus \{s_h\}$ and $t(j) = y$. Since X is ℓ-extendable, this tuple is unique. Therefore $x_h = x'_h$ and L_j is a Latin cube. $\qquad\square$

A set $L_1, \ldots, L_k$ of ℓ-dimensional Latin cubes of order n is said to be *mutually invertible* if

$$\{(L_1(x_1, \ldots, x_\ell), \ldots, L_k(x_1, \ldots, x_\ell), x_1, \ldots, x_\ell) : (x_1, \ldots, x_\ell) \in [n]^\ell\}$$

is ℓ-extendable.

Proposition 5. *Every set $L_1, \ldots, L_k$ of mutually invertible ℓ-dimensional Latin cubes is mutually orthogonal.*

Proof. Let $s_1 < s_2 < \cdots < s_\ell$ with $s_i \in [k]$ and let $(y_1, \ldots, y_\ell) \in [n]^\ell$. It remains to show that there exists a unique $(x_1, \ldots, x_\ell) \in [n]^d$ such that

$$(L_{s_1}(x_1, \ldots, x_\ell), \ldots, L_{s_\ell}(x_1, \ldots, x_\ell)) = (y_1, \ldots, y_\ell) \ .$$

This follows from the fact that

$$\{(L_1(x_1, \ldots, x_\ell), \ldots, L_k(x_1, \ldots, x_\ell), x_1, \ldots, x_\ell) : (x_1, \ldots, x_\ell) \in [n]^\ell\}$$

is ℓ-extendable. $\qquad\square$

In the case of 2-dimensional Latin cubes, mutual orthogonality is equivalent to mutual invertibility.

Proposition 6. *Every set $L_1, \ldots, L_k$ of mutually orthogonal Latin squares is mutually invertible.*

Proof. We prove that

$$X := \{(L_1(x, y), \ldots, L_k(x, y), x, y) : (x, y) \in [n]^2\}$$

is 2-extendable. Let $z_1, z_2 \in [k+2]$ with $z_1 < z_2$. We claim that for each $(x_1, x_2) \in [n]$ there is a unique tuple $t \in X$ such that $t(z_1) = x_1$ and $t(z_2) = x_2$. Consider the following cases.

- $z_1 = k+1$ and $z_2 = k+2$: The claim immediately follows from the definition of X.
- $z_1 \leq k$ and $z_2 \in \{k+1, k+2\}$: The claim follows from the fact that L_{z_1} is a Latin square.

- $z_1 \leq k$ and $z_2 \leq k$: The claim follows from the fact that L_{z_1} and L_{z_2} are orthogonal.

Therefore X is 2-extendable and $L_1, \ldots, L_k$ is a set of mutually invertible Latin squares. $\qquad\square$

Theorem 7. *$G(k, n)$ has an ℓ-decomposition if and only if there are $k - \ell$ mutually invertible Latin ℓ-dimensional cubes of order n.*

Proof. ($\Longleftarrow$) Let $L_1, \ldots, L_{k-\ell}$ be $k - \ell$ mutually invertible ℓ-dimensional Latin cubes of order n. For each $(x_1, \ldots, x_\ell) \in [n]^\ell$, let $K(x_1, \ldots, x_\ell)$ be the copy of K_k defined by the vector $(v_1, \ldots, v_{k-\ell}, x_1, \ldots, x_\ell)$ where $v_i := L_i(x_1, \ldots, x_\ell)$. This defines n^ℓ copies of K_k. That each copy of K_ℓ in $G(k, n)$ is in one such copy of K_k follows immediately from the fact that

$$\{(v_1, \ldots, v_{k-\ell}, x_1, \ldots, x_\ell) : (x_1, \ldots, x_\ell) \in [n]^\ell\}$$

is ℓ-extendable.

($\Longrightarrow$) Consider an ℓ-decomposition X of $G(k, n)$. Thus X is a set of copies of K_k in $G(k, n)$ such that each copy of K_ℓ is in exactly one copy of K_k in X. Consider each copy of K_k in X to be a k-tuple in $[n]^k$. We now show that X is ℓ-extendable. Let $s_1 < \cdots < s_\ell$ be elements of $[k]$ and $(x_1, \ldots, x_\ell) \in [n]^\ell$. There is a unique tuple $(t_1, \ldots, t_k)$ in X containing the copy of K_ℓ with vertex set $\{(s_1, x_1), \ldots, (s_\ell, x_\ell)\}$. Thus $t(s_i) = x_i$ for all $i \in [\ell]$. Therefore X is ℓ-extendable. By Lemma 4, we obtain $k - \ell$ mutually invertible Latin cubes. $\qquad\square$

Note that Proposition 6 and Theorem 7 provide a long-winded proof of Theorem 1.

4. CONSTRUCTION OF AN ℓ-DECOMPOSITION

This section describes a construction of an ℓ-decomposition.

Lemma 8. *If $n \geq k \geq \ell \geq 2$ and n is prime, then $G(k, n)$ has an ℓ-decomposition.*

Proof. Given $(a_1, \ldots, a_\ell) \in [n]^\ell$, let $K(a_1, \ldots, a_\ell)$ be the set of vertices

$$K(a_1, \ldots, a_\ell) := \left\{ \left(c, \left(\sum_{j=0}^{\ell-1} c^j a_j \right) \bmod n \right) : c \in [k] \right\}$$

in $G(k, n)$. Observe that $K(a_1, \ldots, a_\ell)$ induces a copy of K_k in $G(k, n)$, and we have n^ℓ such copies. We claim that each copy of K_ℓ is in some $K(a_1, \ldots, a_\ell)$. Let $S = \{(c_i, v_i) : i \in [\ell]\}$ be a set of vertices inducing K_ℓ. Thus $c_i \neq c_j$ for all $i \neq j$. We need to show that $S \subseteq K(a_1, \ldots, a_\ell)$ for some $a_1, \ldots, a_\ell$. That is, for all $i \in [\ell]$,

$$\sum_{j=0}^{\ell-1} c_i^j a_j \equiv v_i \pmod{n} .$$

Equivalently,

$$
(2) \quad
\begin{bmatrix}
1 & c_1 & c_1^2 & \dots & c_1^{\ell-1} \\
1 & c_2 & c_2^2 & \dots & c_2^{\ell-1} \\
  &   & \vdots &   &   \\
1 & c_\ell & c_\ell^2 & \dots & c_\ell^{\ell-1}
\end{bmatrix}
\begin{bmatrix}
a_1 \\ a_2 \\ \vdots \\ a_\ell
\end{bmatrix}
\equiv
\begin{bmatrix}
v_1 \\ v_2 \\ \vdots \\ v_\ell
\end{bmatrix}
\quad (\mathrm{mod}\ n) \ .
$$

This $\ell \times \ell$ matrix is a Vandermonde matrix, which has non-zero determinant

$$
\prod_{1 \le i < j \le \ell} (c_i - c_j) \ .
$$

Since $c_i \neq c_j$ and n is a prime greater than any $c_i - c_j$, this determinant is non-zero modulo n. (This trick of taking a Vandermonde matrix modulo a prime is well known, and at least dates to a 1951 construction by Erdős [15] for the no-three-in-line problem.) Thus in the vector space $\mathbb{Z}_n^\ell$ (over the finite field $\mathbb{Z}_n$), the row-vectors of this matrix are linearly independent and (2) has a solution. That is, $S \subseteq K(a_1, \dots, a_\ell)$ for some $a_1, \dots, a_\ell$. $\square$

The next lemma is analogous to a Kronecker product of Latin squares.

Lemma 9. *For all integers $k \ge \ell \ge 1$ and $p, q \ge 1$, if both $G(k, p)$ and $G(k, q)$ have ℓ-decompositions, then $G(k, pq)$ has an ℓ-decomposition.*

Proof. Let $X_1, \dots, X_{p^\ell}$ be the vertex sets of copies of K_k in $G(k, p)$ such that each K_ℓ subgraph appears in exactly one copy. Similarly, let $Y_1, \dots, Y_{q^\ell}$ be the vertex sets of copies of K_k in $G(k, q)$ such that each K_ℓ subgraph of $G(k, q)$ appears in exactly one copy. For $a \in [p^\ell]$ and $b \in [q^\ell]$, if $X_a = \{(i, v_i) : i \in [k]\}$ and $Y_b = \{(i, w_i) : i \in [k]\}$, then let $Z_{a,b}$ be the set of vertices $\{(i, (w_i - 1)p + v_i) : i \in [k]\}$ in $G(k, pq)$. Thus $Z_{a,b}$ induces a copy of K_k.

Let $S = \{(c_i, u_i) : i \in [\ell]\}$ be a set of vertices inducing a K_ℓ in $G(k, pq)$. Say $u_i = (w_i - 1)p + v_i$ where $v_i \in [p]$ and $w_i \in [q]$. Since $\{(c_i, v_i) : i \in [k]\}$ induces K_ℓ in $G(k, p)$, some K_a contains $\{(c_i, v_i) : i \in [k]\}$. Similarly, some K_b contains $\{(c_i, w_i) : i \in [k]\}$. By construction, $S \subseteq Z_{a,b}$. Hence the $Z_{a,b}$ are the vertex sets of copies of K_k in $G(k, pq)$ such that each K_ℓ subgraph of $G(k, pq)$ appears in some copy. There are $(pq)^\ell$ such sets $Z_{a,b}$. Thus the $Z_{a,b}$ are an ℓ-decomposition of $G(k, pq)$. $\square$

Lemmas 8 and 9 imply the following, which is one of the main results of the paper.

Theorem 10. *If $n \ge k \ge \ell \ge 2$ and no prime less than k divides n, then $G(k, n)$ has an ℓ-decomposition.*

Theorems 7 and 10 imply:

Theorem 11. *If $n \ge k \ge \ell \ge 2$ and no prime less than k divides n, then there exists a set of $k - \ell$ mutually invertible ℓ-dimensional Latin cubes.*

To generalise the above results, consider the following definition. For integers $k \geq \ell \geq 1$ and $n \geq 1$, let $f(k, n, \ell)$ be the minimum number of copies of K_k in $G(k, n)$ such that each K_ℓ subgraph of $G(k, n)$ appears in some copy. Note that $f(k, n, \ell) \geq n^\ell$ because no two of the n^ℓ copies of K_ℓ that are contained in the first ℓ colours classes of $G(k, n)$ are contained in a single copy of K_k. And $f(k, n, \ell) = n^\ell$ if and only if $G(k, n)$ has an ℓ-decomposition.

Lemma 12. *For all n and all k, there is an integer n' such that $n \leq n' \leq n + e^{k+o(k)}$ and no prime less than k divides n'.*

Proof. Let p be the product of all primes less than k. Let n' be the minimum integer such that $n' \geq n$ and $n' \equiv 1 \pmod{p}$. Thus $n' \leq n+p$ and no prime less than k divides n'. By the asymptotics of primorials, $p \leq e^{k+o(k)}$; see [14]. The result follows. $\square$

Theorem 10 and Lemma 12 imply that $f(k, n, \ell)$ is never much more than n^ℓ.

Theorem 13. *For fixed $k \geq \ell \geq 1$ and $n \geq 1$,*

$$f(k, n, \ell) \leq n^\ell + O(n^{\ell-1}) \ .$$

Finally we mention that Theorem 13 with $k = 6$ and $\ell = 3$ was recently applied to a problem in combinatorial geometry [16]. Indeed, this problem instigated our research.

5. Note

After submitting this paper we discovered that much of it is known in the literature on "orthogonal arrays" and "covering arrays". An ℓ-extendable set of k-tuples in $[n]^k$ is the set of columns of an orthogonal array with k constraints, n levels and strength ℓ (see [17]), and $f(k, n, \ell)$ is the covering array number $\mathrm{CAN}(\ell, k, n)$ (see [12]). See [7, 10, 11] for some of the seminal results on orthogonal arrays, and see [12, 18] for more recent surveys. Our Lemma 8 is Theorem 3.2 in [18], our Lemma 9 is Theorem 3.4 in [18], and our Theorem 10 is Corollary 3.5 in [18]. Other results in this paper are probably previously known.

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